Role of semiclassical description in the quantumlike theory of light rays

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An alternative procedure to the one by Gloge and Marcuse [J. Opt. Soc. Am. **59**, 1629 (1969)] for performing the transition from geometrical optics to wave optics in the paraxial approximation is presented. This is done by employing a recent "deformation" method used to give a quantumlike phase-space description of charged-particle-beam transport in the semiclassical approximation. By taking into account the uncertainty relation (diffraction limit) that holds between the transverse-beam-spot size and the rms of the light-ray slopes, the classical phase-space equation for light rays is deformed into a von Neumann–like equation that governs the phase-space description of the beam transport in the semiclassical approximation. Here, $\hbar$ and the time are replaced by the inverse of the wave number, $\lambdabar$, and the propagation coordinate, respectively. In this framework, the corresponding Wigner-like picture is given and the quantumlike corrections for an arbitrary refractive index are considered. In particular, it is shown that the paraxial-radiation-beam transport can also be described in terms of a fluid motion equation, where the pressure term is replaced by a quantumlike potential in the semiclassical approximation that accounts for the diffraction of the beam. Finally, a comparison of this fluid model with Madelung's fluid model is made, and the classical-like picture given by the tomographic approach to radiation beams is advanced as a future perspective. [S1063-651X(99)18110-8]

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I. MUTUAL CONNECTION OF OPTICS AND MECHANICS

Thirty years ago, Gloge and Marcuse [1] extended the correspondence between optics and mechanics, going from quantum mechanics to wave optics. They performed a transition from geometrical optics to wave optics in a way that was fully similar to the one given to transit from classical mechanics to quantum mechanics. In the formal quantization of Gloge and Marcuse, a set of quantization rules (in which $\hbar$ and the time are replaced by the inverse of the wave number and the propagation coordinate, respectively) is introduced in the Hamiltonian for electromagnetic (e.m.) rays. The result is the e.m. wave equation whose limit, in the paraxial approximation, gives a Schrödinger-like equation called the Fock-Leontovich equation [2]:

$$i\lambdabar\frac{\partial}{\partial z}\Phi=-\frac{\lambdabar^2}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi+U\Phi, \tag{1}$$

where $\lambdabar\equiv\lambda/2\pi$ (λ being the wavelength), z is the propagation coordinate, x and y are the transverse coordinates, and U is an affective dimensionless potential energy proportional to the refractive index n. Equation (1) is an equation for the complex e.m. field amplitude. Provided that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Phi(x,y,z)|^2dx\,dy=1, \tag{2}$$

$|\Phi(x,y,z)|^2$ gives the normalized e.m. power density as well as the probability density of finding an e.m. ray at the transverse location (x,y). Equation (1) is appropriate for describing an e.m. beam traveling along the z axis.

It is worth noting that the limit $\lambdabar\to 0$ recovers the geometrical optics (light-ray equation). In fact, the physical meaning of $\lambdabar$ is given in terms of the diffraction parameter. The condition $\lambdabar\neq 0$ in the paraxial approximation is connected to a weak displacement of light rays from the beam-propagation direction in such a way as to produce a mixing between them. When the beam is traveling *in vacuo*, in the exact geometrical optics limit ($\lambdabar=0$), if the ray slopes are initially all parallel to each other, the ray will be straight-line parallel to the propagation direction. For finite $\lambdabar$, the ray's mixing (diffraction effect) produces a hyperbolic hyperboloid around the z axis that corresponds to a typical caustic shape [3].

The procedure of Gloge and Marcuse proved to be fruitful because it provided for transferring algorithms and many solutions of quantum mechanics to radiation beam physics, especially for optical fibers [4,5], coherent and squeezed state theories [6–11], Schrödinger cat states [12,13], and phase-space investigations within a Wigner-like picture [14] where a quasiclassical distribution, fully similar to the quantum Wigner transform [15] governs the paraxial e.m. ray evolution. At the present time, quantum methodologies are applied to a very large body of branches in which the physics involved is basically classical. Such kinds of descriptions are referred as to *quantumlike descriptions* [16,17].

In this paper, we propose a method, alternative to the one of Gloge and Marcuse, to transit from geometrical optics to wave optics, namely, from the classical-like description to the quantumlike description of light-ray optics, by using a deformation method employed recently in electron optics [18]. This allows us to get an effective description of light-

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ray optics that shows the role played by the semiclassical approximation in the quantumlike theory of light rays. In Sec. II we present the classical-like phase-space equation for light rays for an arbitrary refractive index. In Sec. III the deformation procedure is used to transit from the above classical-like phase-space equation to an effective quantum-like equation in the semiclassical approximation, which formally coincides with the von Neumann equation. This ''deformed'' phase-space description allows us to recover the Wigner-like picture, which is widely used to describe the e.m. beam transport in phase space [14]. The quantumlike picture of Gloge and Marcuse as well as the Fock-Leontovich Schrödinger-like equation are then recovered in the semiclassical approximation. In Sec. IV the hierarchy of the moment equations that are associated with the von Neumann-like equation is obtained, and a fluid model that is associated with the beam transport in real space is derived by truncation of the above hierarchy. In particular, the cases of both the classical and semiclassical fluids are considered. In Sec. V the fluid description is compared with Madelung's fluid model. Finally, in Sec. VI conclusions are summarized, remarks are presented, and future perspectives are discussed, with special attention given to the classical-like picture provided by the tomographic approach to radiation beams.

II. CLASSICAL-LIKE PHASE-SPACE EQUATION FOR LIGHT RAYS

In this section we develop the classical-like description of geometric optics in the phase space in terms of a classical phase-space distribution of the light rays in the case of an arbitrary refractive index. We confine our attention to the case of the paraxial approximation, namely, $dx/dz \equiv x' \ll 1$ and $dy/dz \equiv y' \ll 1$. Taking into account this approximation, it can easily be shown that the following single-light-ray equations hold:

$$\frac{d\vec{r}_\perp}{dz} = \vec{\mathcal{P}}_\perp, \tag{3}$$

$$\frac{d\vec{\mathcal{P}}_\perp}{dz} = -\vec{\nabla}_\perp U, \tag{4}$$

where $\vec{r}_\perp \equiv x\hat{x} + y\hat{y}$, $\mathcal{P}_x \equiv (n/n_0)x'/(1+x'^2+y'^2)^{1/2} \approx (n/n_0)x'$, $\mathcal{P}_y \equiv (n/n_0)y'/(1+x'^2+y'^2)^{1/2} \approx (n/n_0)y'$, and $\vec{\mathcal{P}}_\perp \equiv \mathcal{P}_x\hat{x} + \mathcal{P}_y\hat{y}$ (n_0 being the constant average of n close to the z axis). Thus we can associate with a single ray a classical-like particle trajectory. Consequently, Eq. (4) shows that the refractive index provides an effective force on each single light ray. Consequently, in the paraxial approximation, as for the particle systems, we may introduce a distribution in phase space $\rho(x,y,\mathcal{P}_x,\mathcal{P}_y,z)$ that is constant in its characteristics, i.e.,

$$\frac{\partial\rho}{\partial z} + \{\rho, H\} = 0, \tag{5}$$

where $\{,\}$ denotes the classical Poisson brackets. Equation (5) can be explicitly written as

$$\frac{\partial\rho}{\partial z} + (\vec{\mathcal{P}}_\perp \cdot \vec{\nabla}_\perp)\rho - (\vec{\nabla}_\perp U)\cdot \frac{\partial\rho}{\partial\vec{\mathcal{P}}_\perp} = 0. \tag{6}$$

Let us consider, around the point $(\vec{r}_{\perp 0}, \vec{\mathcal{P}}_{\perp 0})$, the phase-space volume element $d^2 r_\perp d^2\mathcal{P}_\perp = dx\,dy\,d\mathcal{P}_x d\mathcal{P}_y$. Here the quantity $\rho(\vec{r}_{\perp 0}, \vec{\mathcal{P}}_{\perp 0}, z) d^2 r_\perp d^2\mathcal{P}_\perp$ is the probability of finding a light ray at the transverse location $\vec{r}_{\perp 0}$ with slope $\vec{\mathcal{P}}_{\perp 0}$, provided that the following normalization condition holds:

$$\int \rho(\vec{r}_\perp, \vec{\mathcal{P}}_\perp, z) d^2 r_\perp d^2\mathcal{P}_\perp = 1. \tag{7}$$

Equation (6) describes the evolution of the light rays in the paraxial approximation and in the geometrical optics context. However, we point out that Eq. (6) is still suitable to describe the beam optics beyond the exact geometrical-optics limit. In fact, in the case of vacuum, $U=0$, and in the case of linear focusing (defocusing) devices, $U = k_1 x^2/2 + k_2 y^2/2$, Gaussian beams, whose propagation is affected by the diffraction, can be also described by Eq. (6). To give the reader an idea, let us consider a simple two-dimensional (2D) (the y-transverse component is neglected, for simplicity) focusing/defocusing, infinitely thick (in both the x and z directions) device with refractive index of the form $U = k(z)x^2/2$, with $k(z)$ being the strength of the device. Thus in this case Eq. (6) becomes

$$\frac{\partial\rho}{\partial z} + p\frac{\partial\rho}{\partial x} - k(z)x\frac{\partial\rho}{\partial p} = 0, \tag{8}$$

where, for simplicity, we have put $\mathcal{P}_x \equiv p$. We look for a solution of Eq. (8) of the form

$$\rho(x,p,z) = A\exp\left\{-\frac{1}{B}[c(z)x^2 + 2a(z)xp + b(z)p^2]\right\}, \tag{9}$$

where A and B are positive constants and $a(z)$, $b(z)$, $c(z)$ are functions to be determined. Thus, by defining the second-order moments $\sigma_x(z)$, $\sigma_p(z)$, and $\sigma_{xp}(z)$ of ρ as

$$\sigma_x^2(z) = \int \rho(x,p,z)x^2 dx\,dp \equiv \langle p^2\rangle \tag{10}$$

(the beam spot-size),

$$\sigma_p^2(z) = \int \rho(x,p,z)p^2 dx\,dp \equiv \langle p^2\rangle \tag{11}$$

(the momentum spread or rms of ray slopes), and

$$\sigma_{xp}(z) = \int \rho(x,p,z)xp\,dx\,dp \equiv \langle xp\rangle \tag{12}$$

(the ray correlation), Eq. (9) can be cast in the following normalized form:

$$\rho(x,p,z)=\frac{1}{\pi\lambda\!\!\!^{-}}\exp\left\{-\frac{2}{\lambda\!\!\!^{-2}}[\sigma_p^2(z)x^2-2\sigma_{xp}(z)xp+\sigma_x^2(z)p^2]\right\}, \tag{13}$$

where

$$\sigma_x=\lambda\!\!\!^{-} b,\quad \sigma_{xp}=\lambda\!\!\!^{-} a=\sigma_x\frac{d\sigma_x}{dz},\quad \sigma_p=\lambda\!\!\!^{-} c, \tag{14}$$

and

$$B=\frac{\lambda\!\!\!^{-}}{2}. \tag{15}$$

It can be also easily proven that $\sigma_x^2(z)\sigma_p^2(z)-\sigma_{xp}^2(z)=\lambda\!\!\!^{-2}/4=\text{const}$, and consequently,

$$\sigma_x^2(z)\sigma_p^2(z)\geqslant\frac{\lambda\!\!\!^{-}}{2}, \tag{16}$$

which represents a sort of quantumlike uncertainty relation. In particular, the minimum uncertainty corresponds to the well-known diffraction limit [3]

$$(\sigma_x\sigma_p)_{min}\simeq\frac{\lambda\!\!\!^{-}}{2}. \tag{17}$$

Equation (17) is usually observed when the focusing of a monochromatic radiation beam of wavelength λ is produced *in vacuo* [3]. Furthermore, we observe that the above phase-space distribution function of light rays associated with a Gaussian beam gives the following space density:

$$\Lambda_x(x,z)\equiv\int\rho(x,p,z)dp=\frac{1}{\sqrt{2\pi\sigma_x^2(z)}}\exp\left[-\frac{x^2}{2\sigma_x^2(z)}\right]; \tag{18}$$

and the following ray-slope distribution:

$$\Lambda_p(p,z)\equiv\int\rho(x,p,z)dx=\frac{1}{\sqrt{2\pi\sigma_p^2(z)}}\exp\left[-\frac{p^2}{2\sigma_p^2(z)}\right]. \tag{19}$$

It is worthwhile to observe that, in the case where the beam is in a vacuum ($k=0$), solution (9), remains formally the same.

Remarkably, Eqs. (17) and (13) show that, due to the diffraction limit, we cannot resolve among two or more light rays in phase-space regions of size the order of $\lambda\!\!\!^{-}$. If the limit $\lambda\!\!\!^{-}\to 0$ is not exactly taken, but nevertheless $\lambda\!\!\!^{-}$ is considered, however small, we are still within the framework of geometrical optics. Thus in the paraxial approximation Eq. (8) still describes the phase-space evolution in a linear device. On the other hand, the diffraction limit introduced by nonzero $\lambda\!\!\!^{-}$ introduces, by virtue of Eq. (17), an indistinguishability among the light rays.

In the next section, we develop an effective phase-space description that takes into account this indistinguishability. We conclude the present section by observing that $\Lambda_x(x,z)$ must also represent, according to the results of Sec. I, the e.m. power density that is proportional to the modulus square of the e.m. field amplitude associated with the beam.

III. DEFORMED PHASE-SPACE DESCRIPTION

In this section, we apply a deformation method that was used recently in electron optics to transit from the classical phase-space ray equation to a quantumlike phase-space ray equation in the semiclassical approximation [18]. We want to make a similar transition here, starting from the classical phase-space light ray equation (6). We still confine our attention to the 2D case (the y direction is ignored, for simplicity) and take the same steps as in [18]. In spite of the fact that the formalism we apply below is almost identical to the formalism applied for the description of the electronic rays [18], the radiation field treated here is a physically different object; this is the reason we present in detail how to derive the von Newman equation for the phase-space description of light beams, which is a new aspect in comparison with the electronic ray physics of [18].

Let σ_0 be the minimum spot size that can be achieved *in vacuo* with an initial focusing condition, and let us define the parameter $\eta\equiv\lambda\!\!\!^{-}/(2\sigma_0)$. It is easy to see that, in the paraxial approximation, this quantity is much smaller than 1. In fact, by denoting by σ_{p0} the rms of the ray slopes corresponding to the above minimum spot size, from Eq. (17) is clear that

$$\eta\equiv\frac{\lambda\!\!\!^{-}}{2\sigma_0}\simeq\sigma_{p0}\simeq\left\langle\left(\frac{dx}{dz}\right)^2\right\rangle_{max}^{1/2}\ll 1. \tag{20}$$

The above 2D phase-space light-ray equation for an arbitrary refractive index can be explicitly written as

$$\frac{\partial\rho}{\partial z}+p\frac{\partial\rho}{\partial x}-\frac{\partial U}{\partial x}\frac{\partial\rho}{\partial p}=0. \tag{21}$$

By introducing the dimensionless variables

$$\overline{z}\equiv\frac{z}{2\sigma_0},\quad \overline{x}\equiv\frac{x}{2\sigma_0}, \tag{22}$$

Eq. (21) assumes the form

$$\frac{\partial\overline{\rho}}{\partial}\overline{z}+p\frac{\partial\overline{\rho}}{\partial\overline{x}}-\left(\frac{\partial\overline{U}}{\partial\overline{x}}\right)\frac{\partial\overline{\rho}}{\partial p}=0, \tag{23}$$

where $\overline{\rho}\equiv\rho(x/2\sigma_0,p,z/2\sigma_0)\equiv\overline{\rho(\overline{x},p,\overline{z})}$ and $\overline{U}\equiv U(x/2\sigma_0,z/2\sigma_0)\equiv\overline{U}(\overline{x,z})$.

According to the above results, the indistinguishability among two or more rays due to the paraxial diffraction is of the order of $\eta\ll 1$. Thus $\partial\overline{U}/\partial\overline{x}$ in Eq. (21) can be conveniently replaced by the following symmetrized Schwarz-like finite difference ratio:

$$\frac{\partial\overline{U}}{\partial x}\approx\frac{\overline{U}(\overline{x}+\eta/2)-\overline{U}(\overline{x}-\eta/2)}{\eta}. \tag{24}$$

This way, Eq. (21) must be replaced by the following equation for an effective distribution, say $\overline{\rho_w}(\overline{x},p,\overline{z};\eta)$:

$$\frac{\partial\overline{\rho}_w}{\partial\overline{z}}+p\,\frac{\partial\overline{\rho}_w}{\partial\overline{x}}-\frac{\overline{U}(\overline{x}+\eta/2)-\overline{U}(\overline{x}-\eta/2)}{\eta}\,\frac{\partial\overline{\rho}_w}{\partial p}=0. \quad (25)$$

Given the smallness of η, multiplying both the numerator and denominator of the last term of the left-hand side (lhs) by the imaginary unit i, we have

$$\frac{\overline{U}(\overline{x}+\eta/2)-\overline{U}(\overline{x}-\eta/2)}{i\,\eta}\,i\,\frac{\partial\overline{\rho}_w}{\partial p}$$
$$\approx\frac{\overline{U}(\overline{x}+i(\eta/2)\partial/\partial p)-\overline{U}(\overline{x}-i(\eta/2)\partial/\partial p)}{i\,\eta}\,\overline{\rho}_w. \quad (26)$$

Thus, going back to the old variables x and z, Eq. (25) assumes formally the look of a von Neumann equation [15,19]

$$\left\{\frac{\partial}{\partial z}+p\,\frac{\partial}{\partial x}+\frac{i}{\lambdabar}\left[U\left(x+i\,\frac{\lambdabar}{2}\,\frac{\partial}{\partial p}\right)-U\left(x-i\,\frac{\lambdabar}{2}\,\frac{\partial}{\partial p}\right)\right]\right\}\rho_w=0, \quad (27)$$

where $\rho_w\equiv\overline{\rho}_w(2\sigma_0\overline{x},p,2\sigma_0\overline{z};2\sigma_0\eta)\equiv\rho_w(x,p,z;\lambdabar)$. Equation (27) shows that, in the framework of this effective description, the phase-space evolution equation for light rays is a quantumlike phase-space equation, where $\hbar$ and the time t are replaced by $\lambdabar$ and the propagation coordinate z, respectively. However, some considerations are in order.

Approximation (24) is due both to the smallness of η and the fact that an evaluation of $\overline{U}$ variation around the location $\overline{x}$ does not make sense within an interval of size η. This, in fact, corresponds to the intrinsic uncertainty produced among the rays by the paraxial diffraction. Thus, Eq. (25) represents a possible way to take into account the ray mixing produced by the paraxial diffraction in this evaluation.

Since

$$\overline{U}\left(\overline{x}+\frac{i\,\eta}{2}\,\frac{\partial}{\partial p}\right)-\overline{U}\left(\overline{x}-\frac{i\,\eta}{2}\,\frac{\partial}{\partial p}\right)=\frac{\partial\overline{U}}{\partial\overline{x}}\,i\,\eta\,\frac{\partial}{\partial p}+O\left(\eta^3\,\frac{\partial^3}{\partial p^3}\right),$$

approximation (26) is equivalent to assuming that the terms $O(\eta^3\,(\partial^3/\partial p^3))$ are small corrections compared to the lower-order ones, according to the paraxial approximation. Consequently, approximation (26) plays the role of the semiclassical approximation [20].

While the distribution $\rho(x,p,z)$ involved in Eq. (21) is introduced in a classical framework and is positive definite, the function $\rho_w(x,p,z;\lambdabar)$ is introduced in a quantumlike framework and is not positive definite. In fact, in this quantumlike context $\rho_w(x,p,z;\lambdabar)$ cannot be used to give information within the phase-space cells with size smaller than $\lambdabar$, due to the paraxial diffraction, i.e., due to the indistinguishability among the light rays. It is clear from the von Neumann–like equation (27) that ρ_w is a sort of Wigner-like function, which is not positive definite, due to the quantumlike uncertainty principle given in Sec. II. In analogy with quantum mechanics, $\rho_w(x,p,z;\lambdabar)$ can be defined as a quasidistribution, even if its x projection and p projection are actually configuration-space distribution and momentum-space distribution, respectively. In particular, we assume that the probability $\Lambda_x(x,z;\lambdabar)$ introduced above is

$$\Lambda_x(x,z;\lambdabar)=\int\rho_w(x,p,z;\lambdabar)\,dp, \quad (28)$$

provided that ρ_w also is normalized over the phase space.

Remarkably, from the above results it follows that there may exist a complex function, say $\Psi(x,z)$, such that

$$\Lambda_x(x,z;\lambdabar)=\Psi(x,z)\Psi^*(x,z), \quad (29)$$

which is also used for the description of pure quantum states; and the following quantumlike density matrix

$$G(x,x',z)=\Psi(x,z)\Psi^*(x',z), \quad (30)$$

which is also used for the description of mixed quantum states, connected with ρ_w by means of the following Wigner-like transformation:

$$\rho_w(x,p,z;\lambdabar)=\frac{1}{2\pi\lambdabar}\int_{-\infty}^{\infty}G\left(x+\frac{y}{2},x-\frac{y}{2},z\right)\exp\left(i\,\frac{py}{\lambdabar}\right)dy, \quad (31)$$

or, for the pure state,

$$\rho_w(x,p,z;\lambdabar)=\frac{1}{2\pi\lambdabar}\int_{-\infty}^{\infty}\Psi^*\left(x+\frac{y}{2},z\right)\times\Psi\left(x-\frac{y}{2},z\right)\exp\left(i\,\frac{py}{\lambdabar}\right)dy. \quad (32)$$

Consequently, $\Psi(x,z)$ must obey to the following Schrödinger-like equation:

$$i\lambdabar\frac{\partial\Psi}{\partial z}=-\frac{\lambdabar^2}{2}\,\frac{\partial^2}{\partial x^2}\Psi+U(x,z)\Psi, \quad (33)$$

which is exactly the Fock-Leontovich equation in the case of a 2D radiation beam [see Eq. (1)]. Note that Eq. (1) has been recovered by the present deformation method in the semiclassical approximation only. Nevertheless, it is valid, in the paraxial approximation, beyond the semiclassical approximation as well.

IV. CLASSICAL AND SEMICLASSICAL RADIATION FLUIDS

In this section, we consider the hierarchy of moment equations generated by the von Neumann–like equation (27) up to second order. In this way, we can give the picture that we call the ''radiation fluid picture.'' We distinguish the case of $\lambdabar\to 0$ (''classical radiation fluid'') from the one of small wavelengths (''semiclassical radiation fluid''). To this end, one can calculate the set of moment equations associated with Eq. (27), respectively. Defining the following Liouville operator,

$$\hat{\mathcal{L}}\equiv\frac{\partial}{\partial z}+p\,\frac{\partial}{\partial x}-\left(\frac{\partial U}{\partial x}\right)\frac{\partial}{\partial p} \quad (34)$$

(U being an arbitrary refractive index that can be expanded in Taylor series with respect to x), it is easy to see that Eq. (27) can be cast as

$$\hat{\mathcal{L}}\rho_w = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{ƛ}{2}\right)^{2k} \frac{\partial^{2k+1}U}{\partial x^{2k+1}} \frac{\partial^{2k+1}\rho_w}{\partial p^{2k+1}}. \tag{35}$$

Note that Eq. (35) reduces to Eq. (21) when the sum at the rhs is zero. Remarkably, this circumstance is verified not only in the limit $ƛ \to 0$. In fact, it occurs also when, keeping nonzero $ƛ$, the refractive index has a quadratic form in x—this is in full agreement with the results presented in Sec. II. By introducing the ν-order (ν being a non-negative integer) moment of $\hat{\mathcal{L}}$ as

$$\mathcal{M}^{(\nu)}(x,z) \equiv \int_{-\infty}^{\infty} p^{\nu} \hat{\mathcal{L}}\rho_w dp, \tag{36}$$

Eq. (35) leads to the continuity equation for $\nu=0$,

$$\frac{\partial \Lambda_x}{\partial z} + \frac{\partial}{\partial x}(\Lambda_x V) = 0, \tag{37}$$

the motion equation for $\nu=1$,

$$\left(\frac{\partial}{\partial z} + V\frac{\partial}{\partial x}\right)V = -\frac{\partial U}{\partial x} - \frac{1}{\Lambda_x}\frac{\partial \Pi}{\partial x}, \tag{38}$$

the energy equation for $\nu=2$,

$$\frac{\partial u}{\partial z} + \frac{\partial}{\partial x}(uV) + \frac{\partial}{\partial x}(\Pi V) = -\left(\frac{\partial U}{\partial x}\right)\Lambda_x V - \frac{\partial Q}{\partial x}, \tag{39}$$

and so on, where

$$V(x,z) = \frac{1}{\Lambda_x}\int_{-\infty}^{\infty} p\rho_w dp \equiv \langle p \rangle_p \tag{40}$$

is the current velocity, which is experimentally the first-order moment of ρ_w,

$$\Pi(x,z) \equiv \int_{-\infty}^{\infty} (p-V)^2 \rho_w dp \equiv \Lambda_x \langle (p-\langle p\rangle_p)^2 \rangle_p, \tag{41}$$

which is essentially the radiation pressure or the second-order moment of ρ_w,

$$u(x,z) \equiv \frac{1}{2}\Pi + \frac{1}{2}\Lambda_x V^2, \tag{42}$$

$$Q(x,z) \equiv \frac{1}{2}\int_{-\infty}^{\infty} (p-V)^3 \rho_w dp \equiv \Lambda_x \langle (p-\langle p\rangle_p)^3 \rangle_p, \tag{43}$$

which is essentially the analog of the heat. Additionally, from Eq. (36) we also obtain

$$\mathcal{M}^{(\nu)}(x,z) = -\sum_{k=1}^{k_{max} \leqslant (\nu-1)/2} (-1)^k \binom{\nu}{2k+1}\left(\frac{ƛ}{2}\right)^{2k} \times \frac{\partial^{2k+1}U}{\partial x^{2k+1}} \int_{-\infty}^{\infty} p^{\nu-2k-1}\rho_w dp \neq 0, \quad \forall \nu \geqslant 3. \tag{44}$$

The characteristic of these moment equations is that the one that is of ν order is an evolution for the ν-order moment of ρ_w, but contains the $(\nu+1)$-order moment of this function. Provided that a closure equation is introduced, which relates the $(\nu+1)$-order moment with the lower-order ones, the truncated set of equations, consisting of moment equations up to the ν-order plus the closure equation, is fully equivalent or Eq. (35), respectively.

The fluid description is given when the truncation is introduced at $\nu=1$ together with a closure relationship involving the second-order moment. Actually, the picture that we could obtain from the truncation involving Eq. (39) can be considered as a fluid picture as well. Furthermore, note that all of Eqs. (44) account for quantum-like corrections beyond the semiclassical approximation.

We can estimate the order of the paraxial diffraction introduced in Eq. (38), assuming the form (13) for ρ_w and making use of Eqs. (40) and (41). This results in

$$\Pi(x,z) = \frac{ƛ^2}{4\sigma_x^2}\Lambda_x(x,z). \tag{45}$$

Since

$$\frac{ƛ}{2\sigma_x} \leqslant \frac{ƛ}{2\sigma_0} = \eta \ll 1, \tag{46}$$

the last term in Eq. (38), viz.,

$$\frac{1}{\Lambda_x}\frac{\partial \Pi}{\partial x} \sim \frac{ƛ^2}{4\sigma_x^2}\frac{1}{\Lambda_x}\frac{\partial \Lambda_x}{\partial x}, \tag{47}$$

represents the semiclassical approximation of the paraxial diffraction at the level of the fluid description. Remarkably, truncating the hierarchy at the order ν higher and higher, we get a *mesoscopic* description that is deeper and deeper. Taking all the infinite hierarchy, we will have the deepest mesoscopic description of the system (beyond the semiclassical approximation), which corresponds to a fluid scheme that we could call ''Madelung's radiation fluid'' (see Sec. V).

A. Classical radiation fluid (diffractionless beam)

For arbitrary refractive index U, the fluid description for a diffractionless beam can be obtained from Eqs. (37) and (38) in the limit $ƛ \to 0$,

$$\frac{\partial \Lambda_x^{(0)}}{\partial z} + \frac{\partial}{\partial x}(\Lambda_x^{(0)} V^{(0)}) = 0, \tag{48}$$

$$\left(\frac{\partial}{\partial z} + V^{(0)}\frac{\partial}{\partial x}\right)V^{(0)} = -\frac{\partial U}{\partial x}, \tag{49}$$

where the superscript (0) means that we are taking the above limit. In this limit, we observe that

$$\rho_w(x,p,z;ƛ \to 0) \to \Lambda_x^{(0)}(x,z)\,\delta(p - V^{(0)}(x,z)) \equiv \rho_0(x,p,z), \tag{50}$$

and the local slope of the light rays, $p = dx/dz$, is determined only by the gradient of U. In particular, *in vacuo* ($U=0$) a monochromatic beam has the phase-space density of the form $P_0\delta(p-V_0)$, with P_0 and V_0 constants.

Note that system (48),(49) is naturally closed. It has been used in radiation beam optics to solve a number of problems when the diffraction is negligible [21].

B. Semiclassical radiation fluid

Within the fluid description, we now also take into account the paraxial diffraction. Thus, the semiclassical fluid is described by Eqs. (37) and (38) plus a suitable closure equation. The result shown by Eq. (45) for Gaussian beams suggests that we assume, in general, this relationship, at the present level of fluid description, to be the required suitable closure equation. Consequently, Eqs. (37) and (38) become

$$\frac{\partial \Lambda_x^{(1)}}{\partial z}+\frac{\partial}{\partial x}(\Lambda_x^{(1)}V^{(1)})=0, \tag{51}$$

$$\left(\frac{\partial}{\partial z}+V^{(1)}\frac{\partial}{\partial x}\right)V^{(1)}=-\frac{\partial U}{\partial x}-\frac{\lambdabar^2}{4\sigma_x^2}\frac{1}{\Lambda_x^{(1)}}\frac{\partial \Lambda_x^{(1)}}{\partial x}, \tag{52}$$

where the subscript (1) means that the paraxial diffraction is now taken into account. This system is fully similar to the one that can be usually derived for the transverse motion of a dilute particle beam, assuming a fluid model with the ideal gas state equation [22]. In fact, in this analogy, the term $(\lambdabar^2/4\sigma_x^2)(1/\Lambda_x^{(1)})(\partial\Lambda_x^{(1)}/\partial x)$ is replaced, for particle beams, by $(\epsilon^2/4\sigma_x^2)(1/n)(\partial n/\partial x)\simeq(v_{th}^2/c^2)(1/n)(\partial n/\partial x)$, where n is the beam number density, ϵ is the transverse particle beam emittance, v_{th} is the transverse thermal velocity, and the following properties hold: $\epsilon/2\sigma_x\simeq v_{th}/c\ll 1$ (see [23]), and the ideal gas state equation assumed in this case is

$$\Pi=\frac{k_BT}{mc^2}n=\frac{\epsilon^2}{4\sigma_x^2}n\simeq\frac{v_{th}^2}{c^2}n, \tag{53}$$

where here Π plays the role of the transverse kinetic pressure. On the other hand, radiation pressure is the effect that radiation produces on the surroundings (ponderomotive action), which is proportional to the square modulus of the e.m. field amplitude E; i.e., $\Pi\propto|E|^2$. We note that $|E|^2$ and $\Lambda_x^{(1)}$ essentially coincide (apart from a normalization factor); i.e., $|E|^2\propto\Lambda_x^{(1)}$. Consequently, we can provide for the following physical interpretation of the closure equation (45). We observe that since $\sigma_p\sim\lambdabar/2\sigma_x$, the mean transverse energy, due to the diffraction, associated with a single light ray (*in vacuo*) is $\mathcal{E}_0\equiv\frac{1}{2}\sigma_p^2\sim\lambdabar^2/8\sigma_x^2$. We recall that $\Lambda_x(x,z)$ is the probability of finding a light ray at location (x,z). Thus, using arguments analogous to the ones used for particle systems (i.e., electronic-ray systems), we conclude that the transverse radiation pressure is given by

$$\Pi(x,z)=2\Lambda_x^{(1)}(x,z)\mathcal{E}_0(z)\sim\frac{\lambdabar^2}{4\sigma_x^2(z)}\Lambda_x^{(1)}(x,z). \tag{54}$$

C. Coherent states in the semiclassical fluid description

In this section, we give a relevant example of the use of the results presented in the Sec. IV B. In particular, we show that Eqs. (51) and (52) are suitable for describing in a natural way coherent states associated with the radiation fluid motion.

Let us start by considering the case of $V(x,z)$ independent of x, viz.,

$$V(x,z)\equiv p_0(z). \tag{55}$$

In this way, Eq. (52) can be easily integrated with respect to x, giving the normalized density

$$\Lambda_x^{(1)}(x,z)=\frac{\exp\{-(4\sigma_x^2(z)/\lambdabar^2)[U(x,z)+p_0'(z)x+g(z)]\}}{\int_{-\infty}^{\infty}\exp\{-(4\sigma_x(z)/\lambdabar^2)[U(x,z)+p_0'(z)x+g(z)]\}dx}, \tag{56}$$

where $g(z)$ is an arbitrary function of z, and Eq. (51) becomes

$$\frac{\partial\Lambda_x^{(1)}}{\partial z}=-p_0(z)\frac{\partial\Lambda_x^{(1)}}{\partial x}. \tag{57}$$

Note that the density is Gaussian if, and only if, U is quadratic in x. Thus, substituting Eqs. (56) and (57), we obtain

$$\frac{\partial U}{\partial z}+p_0(z)\frac{\partial U}{\partial x}=-p_0''(z)x-p_0(z)p_0'(z)-g(z). \tag{58}$$

Let us define the center $x_0(z)$ of the transverse distribution $\Lambda_x^{(1)}(x,z)$, i.e., the mean value of x

$$x_0(z)\equiv\int_{-\infty}^{\infty}x\Lambda_x^{(1)}(x,z)dx. \tag{59}$$

In general, this quantity could not be zero. Taking into account this observation, we can now define $\sigma_x(z)$ as

$$\sigma_x^2(z)\equiv\int_{-\infty}^{\infty}[x-x_0(z)]^2\Lambda_x^{(1)}(x,z)dx. \tag{60}$$

We now concentrate our attention on the case in which the beam does not spread, namely,

$$\sigma_x(z)\equiv\sigma_{x0}=\text{const.} \tag{61}$$

Thus, by differentiating Eq. (60) with respect to z and taking into account Eqs. (57) and (61), we obtain

$$x_0'(z)=p_0(z). \tag{62}$$

Let us next concentrate our attention only on the case where U is independent of z. In this case, Eq. (58) can be easily integrated with respect to x, giving

$$U(x)=-\frac{1}{2}\frac{p_0''(z)}{p_0(z)}x^2-\frac{1}{p_0(z)}\left(\frac{1}{2}\frac{dp_0^2(z)}{dz}+g'(z)\right)x+G, \tag{63}$$

where G is an arbitrary constant, which, without loss of generality, can be put equal to zero. Consequently, the only possible form of $U(x)$ compatible with Eq. (55) is to be quadratic with respect to x. For instance, by choosing

$$U(x)=\frac{1}{2}kx^2 \quad \text{with} \quad k>0, \tag{64}$$

from Eq. (63) we obtain

$$p_0''+kp_0=0 \tag{65}$$

and

$$g(z)=-\frac{p_0^2}{2}+g_0, \tag{66}$$

where g_0 is an arbitrary constant. On the other hand, in view of Eqs. (58), (64), and (65), Eq. (56) can be cast in the form

$$\Lambda^{(1)}{}_x(x,z)=\sqrt{\frac{k}{2\pi}}\frac{2\sigma_{x0}}{\lambda\!\!\!^{-}}\exp\left[-\frac{2\sigma_{x0}^2k}{\lambda\!\!\!^{-2}}(x-x_0(z))^2\right], \tag{67}$$

with

$$p_0'(z)=-kx_0(z), \quad g(z)=\frac{1}{2}kx_0^2(z). \tag{68}$$

Consequently, combining Eqs. (62), (65), (66), and (68), we obtain

$$\frac{1}{2}p_0^2(z)+\frac{1}{2}kx_0^2(z)=g_0=\text{const} \tag{69}$$

and

$$x_0''+kx_0=0. \tag{70}$$

Finally, by combining Eqs. (60), (61), and (67), we obtain the condition that relates k, $\lambda\!\!\!^{-}$, and σ_{x0},

$$k\sigma_{x0}^4=\frac{\lambda\!\!\!^{-2}}{4}, \tag{71}$$

and $\Lambda_x^{(1)}$ can be written as

$$\Lambda_x^{(1)}(x,z)=\frac{1}{\sqrt{2\pi}\sigma_{x0}}\exp\left[\frac{-(x-x_0(z))^2}{2\sigma_{x0}^2}\right]. \tag{72}$$

We thus can conclude that the distribution (72) with Eqs. (61), (62), (65), (68), (70), and (71), describes a coherent state associated with the semiclassical radiation fluid. Its physical meaning is fully equivalent to the one given in the standard description [6–8]. We would like to point out that the quantum coherent states, which are described by the true Schrödinger equation, are only analogs of the ones described by the Fock-Leontovich equation as in [14]. The quantities $x_0(z)$ and $p_0(z)$ account for the real and imaginary parts of the complex shift α, which generates all coherent states, starting from the ground state of both the quantum [6–8] and quantumlike [24] harmonic oscillator for particle beams,

$$\alpha(z)=\frac{x_0(z)}{2\sigma_{x0}}+i\frac{\sigma_{x0}p_0(z)}{\lambda\!\!\!^{-}}\equiv\alpha_1(z)+i\alpha_2(z). \tag{73}$$

Still keeping U independent of z, we conclude this section by considering the case of the equilibrium states (stationary states) associated with the semiclassical radiation fluid, which corresponds to the case of $x_0=$const. Thus, Eq. (62) gives $p_0=0$ and from Eq. (58) we get $g=$const. Consequently, Eq. (56) gives

$$\Lambda_x^{(1)}(x)=\frac{\exp\left[-\frac{4\sigma_{x0}^2}{\lambda\!\!\!^{-}}U(x)\right]}{\int_{-\infty}^{\infty}\exp\left[-\frac{4\sigma_{x0}^2}{\lambda\!\!\!^{-}}U(x)\right]dx}. \tag{74}$$

Note that Eq. (74) represents a stationary state of the radiation beam for an arbitrary refractive index $U(x)$ in the semiclassical approximation.

V. MADELUNG'S RADIATION FLUIDS

In this section, we give the full quantumlike description of the radiation beam beyond the semiclassical approximation developed in Sec. IV. To this end, let us start from the following eikonal representation of the complex e.m. field amplitude Ψ appearing in Eq. (33),

$$\Psi(x,z)=\Lambda_x^{1/2}(x,z)\exp\left[\frac{i}{\lambda\!\!\!^{-}}\Theta(x,z)\right]. \tag{75}$$

Thus, substituting Eq. (75) in Eq. (33), we obtain the following system of equations:

$$\frac{\partial\Lambda_x}{\partial z}+\frac{\partial}{\partial x}(\Lambda_x v)=0, \tag{76}$$

$$\left(\frac{\partial}{\partial z}+v\frac{\partial}{\partial x}\right)v=-\frac{\partial U}{\partial x}+\frac{\lambda\!\!\!^{-2}}{2}\frac{\partial}{\partial x}\left[\frac{1}{\Lambda_x^{1/2}}\frac{\partial^2\Lambda_x^{1/2}}{\partial x^2}\right], \tag{77}$$

where the current velocity v is now given by

$$v(x,z)=\frac{\partial\Theta(x,z)}{\partial x}. \tag{78}$$

Equations (76) and (77) have been widely used in the literature [25,26] to describe the paraxial propagation of a radiation beam, especially in nonlinear media, where the refractive index depends on Λ_x (i.e., $|\Psi|^2$), which is a functional of Λ_x. Moreover, Eqs. (76) and (77) constitute a closed system and are formally identical to the equations that describe the Madelung fluid [27].

The last term on the rhs of Eq. (77) accounts for the pressure term beyond the semiclassical approximation. If we take for Λ_x the same form as the one given by Eq. (18), the pressure term of Eq. (77) coincides with the one shown in Eq. (47), and thus in this case v coincides with $V^{(1)}$. In fact, the term

$$\frac{\lambda^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\Lambda_x^{1/2}}\frac{\partial^2\Lambda_x^{1/2}}{\partial x^2}\right]$$

becomes

$$-\frac{\lambda^2}{4\sigma_x^2}\frac{1}{\Lambda_x}\frac{\partial\Lambda_x}{\partial x}.$$

One important consequence of this result is that coherent states found in the semiclassical approximation for the semiclassical radiation fluids (see Sec. IV) are exact solutions of the Madelung radiation fluid, as well. However, the stationary states are given for $v(x,z)=p_0(z)\equiv 0$. Thus it follows from Eq. (77) that the square root of the density Λ_x must satisfy the following quantumlike eigenvalue problem associated with the Fock-Lentovich equation:

$$\frac{\lambda^2}{2}\frac{\partial^2\Lambda_x^{1/2}}{\partial x^2}+U(x)\Lambda_x^{1/2}=\mathcal{E}\Lambda_x^{1/2}, \tag{79}$$

where $\mathcal{E}$ is a constant. Stationary states (74) are of course only an approximate solution of Eq. (79) in the semiclassical limit.

Note that the above Madelung's fluid approach gives a classical-like picture of the e.m. radiation beam transport in terms of the modulus and phase of the wave function.

VI. CONCLUSIONS, REMARKS, AND FUTURE PERSPECTIVES

In this paper, we have proposed a deformation procedure, which was recently used to give the quantumlike semiclassical description of the electronic-ray optics [18], to describe in a quantumlike context a transition from geometrical optics to wave optics that is alternative to the one proposed by Gloge and Marcuse [1].

We have given a phase-space description of the geometrical optics in terms of a classical probability density distribution of the light rays for an arbitrary refractive index. In this way, taking into account the quantumlike uncertainty relation (diffraction limit) between the rms transverse ray position σ_x and the rms ray slope σ_p, the above deformation procedure has allowed us to transit to a von Neumann–like equation in the semiclassical approximation that provides for a Wigner-like picture of the radiation beam optics in the paraxial approximation.

In turn, this picture has allowed us to recover, in the semiclassical approximation, the Fock–Leontovich parabolic equation and its Gloge-Marcuse quantumlike interpretation. In this context, the possible negativity of the Wigner-like function has been correctly explained in terms of the above quantumlike uncertainty relation.

We have also determined the hierarchy of the moment equations associated with the von Neumann–like equation, and thus given both the classical and semiclassical radiation fluid descriptions in the paraxial approximation. In particular, the inclusion of the paraxial diffraction in the fluid context, which characterizes the semiclassical radiation fluid, has allowed us to naturally describe the coherent states associated with the radiation beam, whose fluid interpretation is in full agreement with the standard one. Finally, a comparison between the above radiation semiclassical fluid and the Madelung fluid has been given.

A classical-like approach, which was developed recently in quantum mechanics and in quantum optics, is the tomographic one for describing quantum states. As a natural implementation of the classical-like picture given by Madelung's fluid, which will be given in an our future paper, we now briefly discuss this method as applied to the e.m. beam transport.

The above transition to describe light beams in terms of the Wigner function (31) and in terms of the light-beam wave function [see (29)] provides for the possibility of introducing a symplectic tomography transform of the Wigner function and the light-beam wave function ($\lambda=1$)

$$w(X,\mu,\nu,z)=\int \rho_w(x,p,z)\times\exp[-ik(X-\mu q-\nu p)]\frac{dkdqdp}{(2\pi)^2}, \tag{80}$$

$$w(X,\mu,\nu,z)=\frac{1}{2\pi|\nu|}\left|\int \Psi(y,z)\exp\left(\frac{i\mu}{2\nu}y^2-\frac{iX}{\nu}y\right)dy\right|^2. \tag{81}$$

Transform (80) has the inverse

$$\rho_w(x,p,z)=\frac{1}{2\pi}\int w(X,\mu,\nu,z)\times\exp[i(X-\mu q-\nu p)]d\mu d\nu dX. \tag{82}$$

The tomography transform (80) was introduced in quantum optics [28], and its partial case (81) was introduced in analytic signal processing [29] and was applied to the quantum problem of diffraction in time in [30]. Below we discuss the properties of the tomography transform in relation to the quantumlike description of light beams. The function $w(X,\mu,\nu,z)$ is a non-negative function; it is obvious from Eq. (81). This function contains the same information on the light beam as the Wigner function (31), in view of relation (82).

The function (80) has the meaning of the probability distribution function (tomographic probability) for random observable X; for $\mu=1$, $\nu=0$, it reads

$$w(X,1,0,z)=|\Psi(X,z)|^2.$$

Let us consider an initial reference frame in phase space (x,p) of a light beam and introduce other reference frames that are obtained from the initial one by rotation (with an angle φ) and scaling (with a scaling parameter λ). The parameters μ,ν are connected with the rotation and scaling parameters by the following relationships:

$$\mu=e^{\lambda}\cos\varphi, \quad \nu=e^{-\lambda}\sin\varphi. \tag{83}$$

The meaning of tomographic probability is that it is determined as the modulus squared of the light-beam wave function in an ensemble of all reference frames in phase space obtained from the initial reference frame by rotation and scaling. Due to this, tomographic probability depends on parameters of the reference frame of the ensemble (83). As follows from Eqs. (80) and (82), tomographic probability is related to the Wigner-like function by an invertable integral transform, is a homogenious function [31]

$$w(aX,a\mu,a\nu,z)=\frac{1}{|a|}w(X,\mu,\nu,z), \tag{84}$$

and is normalized if the light-beam wave function is normalized,

$$\int w(X,\mu,\nu,z)dX=\int |\Psi(x,z)|^2 dx=1.$$

Thus, for the light-beam description one can use tomographic probability alternatively to the light-beam wave function or Wigner quasidistribution function. Due to the dependence on extra parameters, tomographic probability contains the information on the profiles of the modulus of light-beam wave function in all reference frames in the light-ray phase space obtained by the linear canonical transform (rotation and scaling) from the initial reference frame.

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